5. How May Our Present Dilemma Regarding α be Resolved?

What experimental methods offer promise at the present time of resolving the dilemma regarding the two discrepant values of α? The hf splitting in muonium of HUGHES [5] can probably be refined and made more accurate when more intense sources of muons can be made available. The fs splitting in H or D by the new technique of ROBISCOE [6] can be improved as to accuracy. Perhaps the most promising approach, in the opinion of LAMB, JR., however, is the method of CRANE and WILKINSON [23] in which the electron magnetic moment anomaly, $\mu_e/\mu_0 - 1$, is measured directly. To compute α with the required precision from such a measurement, the quantum electro-dynamically derived theoretical formula for μ_e/μ_0 carried out to include the sixth order term (the term in α^3/π^3) is needed. Fortunately a promising start has just been made in the direction of estimating the coefficient of this term by DRELL and PAGELS [24]. Other less direct means of resolving the α-dilemma requiring highly precise measurements in the field of cryogenics are discussed in the subsection, 2.4 (13) entitled "Fluxoid Quantization" in reference 4.

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A Note on the Theory of Decaying Systems and the Computation of Decay Rates

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Dedicated to Prof. J. MATTAUCH *on his 70th birthday*

The theory of decaying systems of FRANZ [1] — which seems finally to settle the basic questions left open so far — has been reformulated assuming a slightly altered initial situation. The currently used formulae for the decay rates (see MANG [2]) have been derived as an approximation to this practically rigorous theory.

The long standing problem of how to describe instable systems recently has found a surprisingly simple and probably final answer in a paper by FRANZ [1]. By carefully studying the actual situation at the beginning of the decay rate measurement, which implies that there must be a macroscopic time for the preparation of radioactive systems, he was able to avoid the drawbacks of the current theories: In the older formulations one has used complex energies and unnormalizable wave functions, not quite understanding how they come in. Although it has been clear for a long time that the poles of the S matrix (or the wave functions) are related to quasistable (resonant) states, it was not clear why these unphysical states had to be used. Also suitable initial wave packets — within reasonable limits — yield exponential decay laws. Others do not. Thus, it still remained the question to be settled why nature or man always seem to produce just the initial situation which at the end leads to exponential decay. Also in ROSENFELD's recent brillant paper [3] this question has not been asked. Apparently, there must be a common reason yielding the same decay law, however man or nature has prepared the system. What is common to all those physical systems is that production and measurement are macroscopic manipulations, and thus it for instance is impossible to fix the time of the beginning of the measurement microscopically. To have seen this is the merit of FRANZ. Fortunately, the theory resulting from this idea is very simple and thus also in this respect is superior to other ways to attack this problem. It is gratifying that not only can one understand the cause for the exponential decay law, one also may search for reasons for deviations from it. Such deviations always occur at very large times. Also some of the conditions derived and used below may be violated. This may be of some relevance to the

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[2] H. J. MANG, Ann. Rev. Nucl. Sci. **14**, 1 [1964].

[3] L. ROSENFELD, Nucl. Phys. **70**, 1 [1965].

so called T violation in the K-meson decay, although the author is not competent to judge this. Its should be said that this theory — although having the nonrelativistic case under discussion — probably is valid in high energy physics as well: Since mathematically only well known analytic properties of wave functions enter, it is much more general than one might think.

It also is gratifying that one easily can obtain a rigorous formula for the decay constant. From this one may derive approximate expressions, which can be compared with the current theories of α decay of THOMAS [4] and MANG [2]. This will be done in the last chapter. In the first one, the assumptions made by FRANZ will be modified, showing that the essential idea is the occurrence of the finite macroscopic preparation time whereas one has some freedom in chosing the initial situation. The theory in this form will be somewhat simpler, although the results are equivalent in both cases.

§ 1. The Decay Law

FRANZ shows that one obtains the exponential decay law if one assumes that one knows that the ensemble consists only of undecayed systems for a finite macroscopic time. He describes the situation like this: All one is sure about is that the α particles are inside of the nucleus at times $-T$ and 0. Actually, his proof can be applied to a more general situation: Even if one knows more, e. g. how the radioactive nuclei have been built up (either in nature on a cosmological scale or in the laboratory by extracting the radioactive substance) his proof holds, since it depends only on the assumption that all the wave functions contributing to the statistical operator (= wave functions occuring in the ensemble) describe α particles in the nucleus (coordinates $|\boldsymbol{x}| \leqq$ nuclear radius R).

In this note, it shall be emphasized that there is a still simpler assumption which leads to the same result, showing that the essential ingredient is not a definite assumption of the knowledge about the ensemble, but just the idea of macroscopic time.

That there may be this other way to understand the decay, can be seen as follows: At a time $t=-T$ the system may be described by a statistical operator $W(-T)$. That is to say: The ensemble may consist of many systems, hopefully mostly in undecayed states. It is not essential, however, that one has achieved this very well since one may pay attention only to the fraction of radioactive systems. How well, naturally, will depend on the swiftness with which man or nature has suceeded in producing radioactive material (or the knowledge that it is radioactive) as compared to the decay time. At time $t=0$ the ensemble has been changed to $W(0)$. This time T is the time elapsed between stopping of the production process and starting of the measurement. It is a macroscopic time (microseconds or so would not matter). Moreover, it is not well defined: During the production one should take care not to measure the decay, of course. Then, it will require a macroscopic time to finish the macroscopic production and the same is true for the beginning of the measurement. Clearly, this is a complicated process, details of which should not matter for the outcome. One may average a bit about the time T. This has no influence on the results, however, and thus will not be done here. The upshot of this is: Although there exists a not well defined macroscopic time T, the decay law should not depend on it.

It is crucial, now, to understand what can be measured beginning after the time T. Apparently, all one can do is to determine the ratio $P(t)$ of the number of decayed systems at time $t \geqq 0$ to the undecayed ones *at the time* $t=0$. If $N(t)$ is the number of undecayed systems, this ratio is given by

$$P(t)=\frac{N(t)}{N(0)}=\frac{p(t)}{p(0)} \tag{1}$$

$$\text{where } p(t)=\frac{N(t)}{N(-T)}\;\frac{\int^{R} \mathrm{d}^3x\,\langle x|\,e^{-iH(t+T)}\,W(-T)\,e^{iH(t+T)}\,|x\rangle}{\mathrm{Tr}[W(-T)]} \tag{2}$$

(putting for a while $\hbar=1$).

Thus:

$$P(t)=\frac{\int^{R} \mathrm{d}^3x\,\langle x|\,e^{-iH(t+T)}\,W(-T)\,e^{iH(t+T)}\,|x\rangle}{\int^{R} \mathrm{d}^3x\,\langle x|\,e^{-iHT}\,W(-T)\,e^{iHT}\,|x\rangle}. \tag{3}$$

The special case that at $t=-T$ one has a state ψ_{-T} corresponds to $W(-T)=|\psi_{-T}\rangle\langle\psi_{-T}|$ and one has very familiar expressions. Assuming that there is no

[4] R. G. THOMAS, Progr. Theoret. Phys. **26**, 667 [1958].

bound state, (3) can be evaluated by applying the same techniques as used by FRANZ: Inserting the operation $\int dE\,|E\rangle\langle E|$ twice, transforming the integrations into $\int^{i\infty} dE$ and $\int^{-i\infty} dE$ along the imaginary axis in the energy plane. Then there are left some residues from the poles at $G_i = E_i - i\,\Gamma_i/2$ and $G_i^* = E_i + i\,\Gamma_i/2$ plus a double integral along the imaginary axis and two mixed terms. The last ones oscillate out in any macroscopic time interval. The double integrals are very small because of the uncertainty principle: it allows no localization in a region smaller than

$$\Delta x \equiv \sqrt{\frac{\hbar}{2\,m}\,T} \sim 10^{-2}\ \text{cm}\ \sqrt{T}\,(\text{sec})\,. \qquad (4)$$

Thus, there is practically no restriction on T (in a macroscopic sense) since $R \sim 10^{-12}$ cm. (In extremely weakly 'bound' systems (4) may be violated, however). By computing these integrals using the asymptotic wave functions FRANZ was able to show that these terms become important at times $\Gamma t > 100$ (and then behave like t^{-2l-3}; l is the angular momentum quantum number). This agrees with the estimate (4), that they are negligible for all practical purposes. The same result had been found before (see ROSENFELD [3]). What remains, then, are the contributions from the residues. Thus, one has

$$P(t) = \frac{\sum_i P_i(0)\, e^{-\Gamma_i t}}{\sum_k P_k(0)} \qquad (5)$$

with

$$P_i(0) = \int^R d^3x\, \text{Res}[\langle x\,|\,G_i\rangle]\,\langle G_i\,|\,W(-T)\,|\,G_i^*\rangle\ \text{Res}[\langle G_i^*\,|\,x\rangle]\ e^{-\Gamma_i T}\,.$$

Or, if only one pole contributes,

$$P(t) = e^{-\Gamma t}, \qquad (6)$$

as desired. Here the $\langle G_i\,|\,x\rangle$ are analytic continuations of the wave functions $\langle E\,|\,x\rangle$ at the poles G_i, $\text{Res}[\langle x\,|\,G_i\rangle]$ their residues. Similar expressions result from FRANZ's assumptions.

So far, it has been assumed that there are no bound states. Inclusion of the latter does not cause trouble: Because of the occurrence of rapidly oscillating functions in the integrals no mixed terms survive. For one bound state $E = -B$ and one complex pole G one obtains

$$P(t) = \frac{\int^R d^3x\,\langle x\,|\,-B\rangle\langle -B\,|\,W\,|\,-B\rangle\langle -B\,|\,x\rangle + \int^R d^3x\,\text{Res}[\langle x\,|\,G\rangle]\,\langle G\,|\,W\,|\,G^*\rangle\,\text{Res}[\langle G^*\,|\,x\rangle]\ e^{-\Gamma(t+T)}}{\int^R d^3x\,\langle x\,|\,-B\rangle\langle -B\,|\,W\,|\,-B\rangle\langle -B\,|\,x\rangle + \int^R d^3x\,\text{Res}[\langle x\,|\,G\rangle]\,\langle G\,|\,W\,|\,G^*\rangle\,\text{Res}[\langle G^*\,|\,x\rangle]\ e^{-\Gamma T}}\,. \qquad (7)$$

It yields the right behaviour in the two limiting cases that one can safely assume that exclusively the ground state is there ($W = |-B\rangle\langle -B|$) or that it definitely is not there:

$$W = (1 - |1-B\rangle)\,(1 - \langle 1-B|)\,.$$

The latter case is compatible approximately with the assumption that the systems are not decayed ($W = \int^R dx\,|x\rangle\langle x|$). Pure exponential decay from excited states may be observed only if the experimentalist has made sure (or almost sure) that the ensemble of undecayed systems has no ground state components. Here it comes in what has been said above, namely that it sometimes is advantageous to use a more general W: It approximately may be chosen such that it has no bound state components and yet describes undecayed systems.

§ 2. The Decay Constant

In general it will be difficult to compute the half life. In the one body theory – dealt with in the foregoing chapter – one may be able to do this numerically or using a model. At present, it is impossible to determine it in the realistic case, at least as long as there is no true many body theory. However, one may find a relation between the wave functions and Γ which may be used for some iteration procedure. The first step is the theory of THOMAS [4] and MANG [2], if one goes over to a many body picture.

Writing (3) as

$$P(t) = \frac{\int^R d^3x \int^\infty dE \int^\infty dE'\langle x\,t\,|\,E\rangle\langle E\,|\,W(-T)\,|\,E'\rangle\langle E'\,|\,x\,t\rangle}{N(0)} \qquad (8)$$

and using the continuity relation

$$\frac{\partial}{\partial t}\langle x\,t\,|\,E\rangle\langle E'\,|\,x\,t\rangle = -\frac{\partial}{\partial x}\frac{\hbar}{2\,m\,i}\left\{\langle E'\,|\,x\,t\rangle\left[\overrightarrow{\frac{\partial}{\partial x}} - \overleftarrow{\frac{\partial}{\partial x}}\right]\langle x\,t\,|\,E\rangle\right. \qquad (9)$$

one obtains

$$\Gamma = -\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = +\frac{\hbar}{2\,m\,i}R^2 \frac{\int\limits_R \mathrm{d}\Omega \int \mathrm{d}E\,\mathrm{d}E' \left\{\langle R\,\Omega\,t\,|\,E\rangle \left[\overleftarrow{\frac{\partial}{\partial r}} - \overrightarrow{\frac{\partial}{\partial r}}\right]\langle E'\,|\,R\,\Omega\,t\rangle\,\langle E\,|\,W(-T)\,|\,E'\rangle\right.}{\int \mathrm{d}^3x \int \mathrm{d}E\,\mathrm{d}t' \langle x\,t\,|\,E\rangle\,\langle E\,|\,W(-T)\,|\,E'\rangle\,\langle E'\,|\,x\,t\rangle} \;; \tag{10}$$

and finally, applying the same techniques in the complex energy plane as above (assuming that there is only one pole of importance and no bound state)

$$\frac{\Gamma}{\hbar} = = -\frac{\hbar}{2\,m\,i}R^2 \frac{\int\limits_R \mathrm{d}\Omega \left\{\langle R\,\Omega\,|\,G\rangle\,\frac{\partial}{\partial r}\,\mathrm{Res}[\langle G^*\,|\,R\,\Omega\rangle] - \frac{\partial}{\partial r}\mathrm{Res}[\langle R\,\Omega\,|\,G\rangle]\,\langle G^*\,|\,R\,\Omega\rangle\right\}}{\int \mathrm{d}^3x\;\mathrm{Res}[\langle x\,|\,G\rangle]\,\mathrm{Res}[\langle G^*\,|\,x\rangle]}. \tag{11}$$

Here $\mathrm{d}\Omega$ means integration over the surface at the radius R. If one choses R large such that one may use asymptotic wave functions

$$\frac{\partial}{\partial r}\,\langle r\,\Omega\,|\,G\rangle \to i\,k\langle r\,\Omega\,|\,G\rangle$$

one obtains

$$\frac{\Gamma}{\hbar} = \frac{\hbar\,(k+k^*)\;R^2}{2\,m}\,\frac{\int\limits_R \mathrm{d}\Omega\,\langle R\,\Omega\,|\,G\rangle\,\langle G^*\,|\,R\,\Omega\rangle}{\int \mathrm{d}^3x\,\langle x\,|\,G\rangle\,\langle G^*\,|\,x\rangle}. \tag{12}$$

Of course, $R \ll \Delta x$ of (4) has to be chosen. Then (12) practically is a rigorous equation. The Res-symbols have been omitted since they affect only a factor in the wave functions occuring in numerator and denominator.

k is the wave number belonging to G:

$$k = k(G), \quad k^2 = 2\,m\,G/\hbar^2, \quad k^* = k(G^*). \tag{13}$$

(12) leads to an iteration procedure for the determination of Γ. For small Γ one can imagine that it may be a good approximation to replace G and G^* by their real part E (decay energy). One is not on very safe ground, however, except that one may replace

$$(k+k^*)/2 \approx \sqrt{2\,m\,E}/\hbar \equiv k_0\,, \tag{14}$$

implying only a small error as long as the system decays slowly (the error is of the order of magnitude 10^{-15} for α decay). Putting the speed of the outcoming particle equal to v, one has

$$\frac{\Gamma}{\hbar} = v\,R^2\,\frac{\int\limits_R \mathrm{d}\Omega\,\langle \Omega\,R\,|\,G\rangle\,\langle G^*\,|\,\Omega\,R\rangle}{\int \mathrm{d}^3x\,\langle\;|\,G\rangle\,\langle G^*\,|\,x\rangle}. \tag{15}$$

As long as R is not too large, such that the exponential increase of the wave functions can be ignored, one may at least in the surface integral replace G and G^* by E. 'R not too large' means

$$2\,R\,\mathrm{Im}\,k \approx \frac{1}{\hbar}\,\frac{\Gamma}{\sqrt{E}}\sqrt{\frac{m}{2}}\,R \ll 1 \tag{16}$$

for $\Gamma \ll E$. This restriction is compatible with the use of the asymptotic wave function. In alpha decay R typically may be restricted to being smaller than 10^6 to 10^{-3} cm. It is harder to say whether in the volume integral one may replace G by the real part E: If one uses the asymptotic wave function everywhere, one obtains a denominator $k - k^*$ which is proportional to Γ.

It is reasonable to introduce new wave functions

$$\langle\langle x\,|\,G\rangle\rangle \equiv \frac{\langle x\,|\,G\rangle}{[\int\limits_R \mathrm{d}^3x \langle x\,|\,G\rangle\langle G^*\,|\,x\rangle]^{1/2}} \tag{17}$$

which are normalized to one in the sphere with the radius R. Thus

$$\frac{\Gamma}{\hbar} = v\,R^2\,\mathrm{d}\Omega\,|\,\langle\langle \Omega\,R\,|\,G\rangle\rangle|^2 \tag{18}$$

(since the relation $\langle G^*\,|\,x\rangle = \langle x\,|\,G^*\rangle$ holds). It should be noted that the same expression follows from FRANZ's assumption about the initial situation. For definite angular momentum l of the outcoming particle, one obtains

$$\Gamma/\hbar = v\,R^2\,|\langle\langle l\,R\,|\,G\rangle\rangle|^2. \tag{19}$$

The current α decay many body theories (see MANG [2]) replace $\langle\langle l\,R\,|\,G\rangle\rangle$ by a new probability amplitude for finding an α particle and daughter nucleus in the parent nucleus:

$$\begin{aligned}\langle\langle l\,R\,|\,G\rangle\rangle &\to (l\,R\,|\,G)\\ &= \left(\frac{N}{2}\right)^{1/2}\left(\frac{Z}{2}\right)^{1/2}\int \mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}\Omega\,\sum_m C(l\,j\,J, m\,M-m)\\ &\quad\cdot Y_l^{m*}(\Omega)\;\psi_j^{*M-m}(\eta)\;\chi^*(\xi)\;\Phi_J{}^M(x_1\ldots x_A)\,.\end{aligned} \tag{20}$$

The normalization of the wave function is important: $\xi_1, \ldots, \xi_3$ and $\eta_1, \ldots, \eta_{A-5}$ are the internal coordinates of α particle and daughter. (20) is assumed to be valid for large distance R of α particle and daughter such that the overlap between the wave functions $\chi(\xi)$ and $\psi_j^m(\eta)$ of α and daughter is negligible. The exclusion principle has been taken into account, since for large R it influences only the normalization. The factors have been chosen such that a wave function of the parent nucleus $\Phi_J^M(x_i)$ which is clustered completely (preformation factor 1)

$$\Phi_J^M(x_i) = \sum_p (-)^p f_l(R) \sum_m C(l\, j\, J, m\, M-m) Y_l^m \psi_j^{M-m} \chi \tag{21}$$

(where $f_l(R)$ describes the radial part of the relative motion of α and daughter and the summation goes over all permutations producing new terms) leads to

$$(x \mid G) = f_l(R)$$

(for large R). The wave functions χ, ψ_j^m and Φ_J^M have been assumed to be antisymmetrized and normalized in terms of ξ, η and ξ, η, R. As long as there is no more rigorous derivation from many body theory, this wave function is the best one has: It yields the same decay rate as the one body theory if one knows that there is one α particle in the nucleus (preformation factor 1), and a smaller decay rate if there is only a finite probability smaller than one for it.

Thus, the current α decay theory appears as an approximation to a practically rigorous one body theory translated into the many body language. How well the approximations (replacing $E - i\Gamma/2$ by E in the wave functions) are justified and whether a better foundation of the many body aspects can be found still remains to be seen.

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